R-MATRICES FOR QUANTUM AFFINE ALGEBRAS AND KHOVANOV-LAUDA-ROUQUIER ALGEBRAS, I

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ABSTRACT. Let I be a finite set of pairs consisting of good $U_q'(\mathfrak{g})$-modules and invertible elements in the base field $\mathbb{C}(q)$. The distribution of poles of normalized R-matrices yields Khovanov-Lauda-Rouquier algebras $R^I(n)$ for $n \geq 0$. We define a functor $\mathcal{F}$ from the category of finite-dimensional $R^I(n)$-modules to the category of finite-dimensional $U_q'(\mathfrak{g})$-modules. We show that the functor $\mathcal{F}$ sends convolution products to tensor products and is exact if $R^I(n)$ is of type A,D,E.

INTRODUCTION

Let $U_q(\mathfrak{g}) = U_q(\widehat{\mathfrak{sl}_N})$ be the quantum affine algebra of type $A^{(1)}_{N-1}$ and let V_{aff} be the affinization of V, the vector representation of $U_q'(\mathfrak{g})$. We denote by H_n^{aff} the affine Hecke algebra. In [4, 5, 8], a functor

$$\mathcal{F}_n \colon H_n^{\mathrm{aff}}\text{-mod} \to U_q'(\mathfrak{g})\text{-mod}$$

is introduced, which is given by

$$M \longmapsto (V_{\mathrm{aff}})^{\otimes n} \otimes_{H_n^{\mathrm{aff}}} M.$$

It was shown that the functor $\mathcal{F}_n$ is exact and that $\mathcal{F}_n$ is fully faithful when $N > n$.

In [18], this idea was generalized to the case when $U_q'(\mathfrak{g})$ is an arbitrary quantum affine algebra and V is a good module. That is, the poles of the R-matrix on $(V_{\mathrm{aff}})^{\otimes n}$ define a quiver, which in turn defines a Khovanov-Lauda-Rouquier algebra $R(n)$. Then one can construct a functor

$$\mathcal{F}_n \colon R(n)\text{-mod} \to U_q'(\mathfrak{g})\text{-mod}$$

given by

$$M \longmapsto \widehat{V}^{\otimes n} \otimes_{R(n)} M,$$

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where $\widehat{V}^{\otimes n}$ is a certain completion of $(V_{\rm aff})^{\otimes n}$. Moreover, it was shown that $\mathcal{F}_n$ sends convolution products to tensor products. When $U'_q(\mathfrak{g}) = U'_q(\widehat{\mathfrak{sl}_N})$ and V is the vector representation, it coincides with the one given in [4].

In this article, we extend this construction to the case when we have a family of good modules. More precisely, let $\{V_s\}_{s\in\mathcal{S}}$ be a family of good $U'_q(\mathfrak{g})$-modules and let I be a finite subset of $\mathcal{S}\times\mathbb{C}(q)^\times$. An element $i\in I$ is denoted by $i=(S(i),X(i))$.

Then we can define a quiver Γ_I with I as a set of vertices as follows. For $i,j\in I$, let $R^{\rm norm}_{V_{S(i)},V_{S(j)}}\colon (V_{S(i)})_u\otimes(V_{S(j)})_v\to(V_{S(j)})_v\otimes(V_{S(i)})_u$ be the normalized R-matrix. We join i and j by edges if $R^{\rm norm}_{V_{S(i)},V_{S(j)}}$ has a pole at $u/v=X(i)/X(j)$. Then the quiver Γ_I defines a Khovanov-Lauda-Rouquier algebra $R^I(n)$. For each sequence $\nu\in I^n$, set

$$V_\nu=(V_{S(\nu_1)})_{\rm aff}\otimes\cdots\otimes(V_{S(\nu_n)})_{\rm aff},$$

and let $\widehat{V}^{\otimes n}$ be a certain completion of $\bigoplus_{\nu\in I^n}V_\nu$. Then $\widehat{V}^{\otimes n}$ has a structure of a $(U'_q(\mathfrak{g}),R^I(n))$-bimodule, and we can define the functor

$$\mathcal{F}_n\colon R^I(n)\text{-mod}\to U'_q(\mathfrak{g})\text{-mod}$$

given by

$$M\longmapsto \widehat{V}^{\otimes n}\otimes_{R^I(n)}M.$$

In this paper, we first prove that $\mathcal{F}_n$ sends convolution products to tensor products (Theorem 2.6). Moreover, when the Cartan datum associated with I is of type A, D, E, we show that $\mathcal{F}_n$ is an exact functor (Theorem 2.9).

In a forthcoming paper, we will study several applications of the above functor $\mathcal{F}_n$. In particular, we will give a construction of categories whose Grothendieck groups have quantum cluster structures. We will also provide an interpretation of the isomorphism between t-deformed Grothendieck rings and negative parts of the quantum groups, which was established recently in [10].

1. Quantum groups and Khovanov-Lauda-Rouquier algebras

1.1. **Quantum groups.** In this section, we recall the definitions of the quantum groups. Let I be a finite index set. A *Cartan datum* is a quintuple $(A,P,\Pi,P^\vee,\Pi^\vee)$ consists of

(a) an integer-valued matrix $A=(a_{ij})_{i,j\in I}$, called the *symmetrizable generalized Cartan matrix*, which satisfies
  (i) $a_{ii}=2$ $(i\in I)$,
  (ii) $a_{ij}\le 0$ $(i\ne j)$,
  (iii) $a_{ij}=0$ if $a_{ji}=0$ $(i,j\in I)$,
  (iv) there exists a diagonal matrix $D=\mathrm{diag}(\mathsf{s}_i\mid i\in I)$ such that DA is symmetric, and s_i are positive integers.

(b) a free abelian group P of finite rank, called the *weight lattice*,

(c) $\Pi = \{\alpha_i \in P \mid \ i \in I\}$, called the set of *simple roots*,
(d) $P^\vee := \mathrm{Hom}(P, \mathbb{Z})$, called the *dual weight lattice*,
(e) $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$, called the set of *simple coroots*,

satisfying the following properties:

(i) $\langle h_i, \alpha_j\rangle = a_{ij}$ for all $i, j \in I$,
(ii) Π is linearly independent,
(iii) for each $i \in I$, there exists $\Lambda_i \in P$ such that $\langle h_j, \Lambda_i\rangle = \delta_{ij}$ for all $j \in I$.

We call Λ_i the *fundamental weights*. The free abelian group $\mathsf{Q} := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ is called the *root lattice*. Set $\mathsf{Q}^+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i \subset \mathsf{Q}$ and $\mathsf{Q}^- = \sum_{i\in I} \mathbb{Z}_{\leq 0}\alpha_i \subset \mathsf{Q}$. For $\beta = \sum_{i\in I} m_i\alpha_i \in \mathsf{Q}$, we set $\mathrm{ht}(\beta) = \sum_{i\in I} |m_i|$.

Set $\mathfrak{h} = \mathbb{Q} \otimes_{\mathbb{Z}} P^\vee$. Then there exists a symmetric bilinear form $(\quad|\quad)$ on $\mathfrak{h}^*$ satisfying

$$(\alpha_i|\alpha_j) = \mathsf{s}_i a_{ij} \quad (i, j \in I) \quad \text{and } \langle h_i, \lambda\rangle = \frac{2(\alpha_i|\lambda)}{(\alpha_i|\alpha_i)} \text{ for any } \lambda \in \mathfrak{h}^* \text{ and } i \in I.$$

Let q be an indeterminate. For each $i \in I$, set $q_i = q^{\mathsf{s}_i}$.

Definition 1.1. *The* quantum group $U_q(\mathfrak{g})$ *associated with a Cartan datum* $(A, P, \Pi, P^\vee, \Pi^\vee)$ *is the associative algebra over* $\mathbb{Q}(q)$ *with* 1 *generated by* e_i, f_i $(i \in I)$ *and* q^h $(h \in P^\vee)$ *satisfying following relations:*

$$\begin{aligned}
&q^0 = 1,\ q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in P^\vee,\\
&q^h e_i q^{-h} = q^{\langle h,\alpha_i\rangle} e_i, \quad q^h f_i q^{-h} = q^{-\langle h,\alpha_i\rangle} f_i \quad \text{for } h \in P^\vee, i \in I,\\
&e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad \text{where } K_i = q^{\mathsf{s}_i h_i},\\
&\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i e_i^{1-a_{ij}-r} e_j e_i^r = 0 \quad \text{if } i \neq j,\\
&\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i f_i^{1-a_{ij}-r} f_j f_i^r = 0 \quad \text{if } i \neq j.
\end{aligned}$$

Here, we set $[n]_i = \dfrac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$, $\quad [n]_i! = \prod_{k=1}^n [k]_i$ and $\begin{bmatrix} m \\ n \end{bmatrix}_i = \dfrac{[m]_i!}{[m-n]_i![n]_i!}$ for each $m, n \in Z_{\geq 0}$, $i \in I$.

We denote by $-$ the $\mathbb{C}$-algebra automorphism of $U_q(\mathfrak{g})$ given by

$$\overline{q} = q^{-1}, \qquad \overline{q^h} = q^{-h} \ \ (h \in P^\vee), \qquad \overline{e_i} = e_i \ (i \in I), \qquad \overline{f_i} = f_i \ (i \in I).$$

We denote by φ and $*$ the $\mathbb{Q}(q)$-algebra anti-automorphisms of $U_q(\mathfrak{g})$ given by

$$(q^h)^* = q^{-h} \quad (h \in P^\vee), \qquad {e_i}^* = e_i \ (i \in I), \qquad {f_i}^* = f_i \ (i \in I),$$
$$\varphi(q^h) = q^h \quad (h \in P^\vee), \qquad \varphi(e_i) = f_i \ (i \in I), \qquad \varphi(f_i) = e_i \ (i \in I).$$

We have two comultiplications $\Delta_\pm \colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ given by

$$\Delta_+(q^h) = q^h \otimes q^h, \quad \Delta_+(e_i) = e_i \otimes 1 + K_i \otimes e_i, \quad \Delta_+(f_i) = f_i \otimes K_i^{-1} + 1 \otimes f_i,$$

$$\Delta_-(q^h) = q^h \otimes q^h, \quad \Delta_-(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \quad \Delta_-(f_i) = f_i \otimes 1 + K_i \otimes f_i.$$

Let $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) be the subalgebra of $U_q(\mathfrak{g})$ generated by e_i's (resp. f_i's), and let $U_q^0(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by q^h $(h \in P^\vee)$. Then we have the *triangular decomposition*

$$U_q(\mathfrak{g}) \simeq U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}),$$

and the *weight space decomposition*

$$U_q(\mathfrak{g}) = \bigoplus_{\beta \in Q} U_q(\mathfrak{g})_\beta,$$

where $U_q(\mathfrak{g})_\beta := \left\{ x \in U_q(\mathfrak{g}) \, ; \, q^h x q^{-h} = q^{\langle h, \beta \rangle} x \text{ for any } h \in P^\vee \right\}$.

Let $\mathbf{A} = \mathbb{Z}[q, q^{-1}]$ and set

$$e_i^{(n)} = e_i^n / [n]_i!, \quad f_i^{(n)} = f_i^n / [n]_i! \ \ (n \in \mathbb{Z}_{\geq 0}).$$

We define the $\mathbf{A}$-form $U_{\mathbf{A}}(\mathfrak{g})$ to be the $\mathbf{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}$, $f_i^{(n)}$ $(i \in I, n \in \mathbb{Z}_{\geq 0})$, q^h $(h \in P^\vee)$. Let $U_{\mathbf{A}}^+(\mathfrak{g})$ (resp. $U_{\mathbf{A}}^-(\mathfrak{g})$) be the $\mathbf{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}$ (resp. $f_i^{(n)}$) for $i \in I$, $n \in \mathbb{Z}_{\geq 0}$.

1.2. **Crystal bases and global bases.**

Let e'_i and e''_i be the operator on $U_q^-(\mathfrak{g})$ defined by

$$[e_i, x] = \frac{e''_i(x) K_i - K_i^{-1} e'_i(x)}{q_i - q_i^{-1}} \quad (x \in U_q^-(\mathfrak{g})).$$

The operator e'_i satisfies

$$e'_i(f_j) = \delta_{ij} \quad (i, j \in I) \text{ and } \quad e'_i(xy) = e'_i(x) y + q_i^{\langle h_i, \mathrm{wt}(x) \rangle} x e'_i(y) \qquad (x, y \in U_q^-(\mathfrak{g})).$$

Hence we have

$$e'_i f_j = q_i^{-\langle h_i, \alpha_j \rangle} f_j e'_i + \delta_{ij},$$

where $f_j \in \mathrm{End}_{\mathbb{Q}(q)}(U_q^-(\mathfrak{g}))$ denotes the left multiplication by $f_j \in U_q^-(\mathfrak{g})$. There exists a unique non-degenerate symmetric bilinear form $(\cdot , \cdot)_-$ on $U_q^-(\mathfrak{g})$ satisfying

$$(1, 1)_- = 1 \text{ and } \quad (e'_i x, y)_- = (x, f_i y)_- \text{ for any } x, y \in U_q^-(\mathfrak{g}).$$

Any element $x \in U_q^-(\mathfrak{g})$ can be uniquely written as

$$x = \sum_{n \geq 0} f_i^{(n)} x_n \quad \text{for some } x_n \in \mathrm{Ker}(e_i').$$

We define the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ on $U_q^-(\mathfrak{g})$ by

$$\tilde{e}_i x = \sum_{n \geq 1} f_i^{(n-1)} x_n, \quad \tilde{f}_i x = \sum_{n \geq 0} f_i^{(n+1)} x_n.$$

Proposition 1.2 ([12]). *Let $\mathbf{A}_0 = \{f \in \mathbb{Q}(q) \mid f$ is regular at $q = 0\}$. Define*

$$L(\infty) = \sum_{\ell \geq 0, i_1, \ldots, i_\ell \in I} \mathbf{A}_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \subset U_q^-(\mathfrak{g}),$$

$$B(\infty) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \ \mathit{mod}\ qL(\infty) \mid \ell \geq 0, i_1, \ldots, i_\ell \in I \ \} \subset L(\infty)/qL(\infty).$$

Then we have

(a) $\tilde{e}_i L(\infty) \subset L(\infty)$ *and* $\tilde{f}_i L(\infty) \subset L(\infty)$,
(b) $B(\infty)$ *is a basis of* $L(\infty)/qL(\infty)$,
(c) $\tilde{f}_i B(\infty) \subset B(\infty)$ *and* $\tilde{e}_i B(\infty) \subset B(\infty) \cup \{0\}$,

We call the pair $(L(\infty), B(\infty))$ the crystal basis of $U_q^-(\mathfrak{g})$.

Set $L(\infty)^- = \{\overline{x} \mid x \in L(\infty)\}$. Then the triple $(L(\infty), L(\infty)^-, U_{\mathbf{A}}^-(\mathfrak{g}))$ is *balanced*; i.e.,

$$L(\infty) \cap L(\infty)^- \cap U_{\mathbf{A}}^-(\mathfrak{g}) \to L(\infty)/qL(\infty)$$

is a $\mathbb{Q}$-vector space isomorphism.

Let G^{low} be the inverse of the above isomorphism. Then

$$\mathbf{B}^{\mathrm{low}} := \left\{ G^{\mathrm{low}}(b) \,;\, b \in B(\infty) \right\}$$

forms a basis of $U_q^-(\mathfrak{g})$. We call it the *lower global basis*. Let

$$\mathbf{B}^{\mathrm{up}} := \{ G^{\mathrm{up}}(b) \,;\, b \in B(\infty) \}$$

be the dual basis of $\mathbf{B}^{\mathrm{low}}$ with respect to the bilinear form $(\cdot, \cdot)_-$. It is called the *upper global basis of* $U_q^-(\mathfrak{g})$.

1.3. **Khovanov-Lauda-Rouquier algebras.**

Now we recall the definition of the Khovanov-Lauda-Rouquier algebras associated with a given Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$.

Let $\mathbf{k}$ be a base field. For $i, j \in I$ such that $i \neq j$, set

$$S_{i,j} = \left\{ (p, q) \in \mathbb{Z}_{\geq 0}^2 \,;\, (\alpha_i|\alpha_i)p + (\alpha_j|\alpha_j)q = -2(\alpha_i|\alpha_j) \right\}.$$

Let us define the polynomials $(Q_{ij})_{i,j\in I}$ in $\mathbf{k}[u,v]$ by

$$Q_{ij}(u,v) = \begin{cases} 0 & \text{if } i=j, \\ \sum_{(p,q)\in S_{i,j}} t_{i,j;p,q} u^p v^q & \text{if } i \neq j. \end{cases} \tag{1.1}$$

They satisfy $t_{i,j;p,q} = t_{j,i;q,p}$ (equivalently, $Q_{i,j}(u,v) = Q_{j,i}(v,u)$) and $t_{i,j:-a_{ij},0} \in \mathbf{k}^{\times}$.

We denote by $S_n = \langle s_1, \dots, s_{n-1}\rangle$ the symmetric group on n letters, where $s_i := (i, i+1)$ is the transposition of i and $i+1$. Then S_n acts on I^n by place permutations.

Definition 1.3. *The* Khovanov-Lauda-Rouquier algebras $R(n)$ of degree n associated with the Cartan datum $(A, P, \Pi, P^{\vee}, \Pi^{\vee})$ and the matrix $(Q_{ij})_{i,j\in I}$ *is the associative algebra over* $\mathbf{k}$ *generated by the elements* $\{e(\nu)\}_{\nu\in I^n}$, $\{x_k\}_{1\le k\le n}$, $\{\tau_m\}_{1\le m\le n-1}$ *satisfying the following defining relations:*

$$\begin{aligned}
& e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu\in I^n} e(\nu) = 1, \\
& x_k x_m = x_m x_k, \quad x_k e(\nu) = e(\nu) x_k, \\
& \tau_m e(\nu) = e(s_m(\nu))\tau_m, \quad \tau_k\tau_m = \tau_m\tau_k \quad \textit{if } |k-m|>1, \\
& \tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})e(\nu), \\
& (\tau_k x_m - x_{s_k(m)}\tau_k)e(\nu) = \begin{cases} -e(\nu) & \textit{if } m=k, \nu_k=\nu_{k+1}, \\ e(\nu) & \textit{if } m=k+1, \nu_k=\nu_{k+1}, \\ 0 & \textit{otherwise}, \end{cases} \\
& (\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) \\
& = \begin{cases} \dfrac{Q_{\nu_k,\nu_{k+1}}(x_k,x_{k+1}) - Q_{\nu_k,\nu_{k+1}}(x_{k+2},x_{k+1})}{x_k - x_{k+2}} e(\nu) & \textit{if } \nu_k = \nu_{k+2}, \\ 0 & \textit{otherwise}. \end{cases}
\end{aligned}$$

The above relations are homogeneous provided with

$$\deg e(\nu) = 0, \quad \deg\ x_k e(\nu) = (\alpha_{\nu_k}|\alpha_{\nu_k}), \quad \deg\ \tau_l e(\nu) = -(\alpha_{\nu_l}|\alpha_{\nu_{l+1}}),$$

and hence $R(n)$ is a ($\mathbb{Z}$-)graded algebra. For a graded $R(\beta)$-module $M = \bigoplus_{k\in\mathbb{Z}} M_k$, we define $qM = \bigoplus_{k\in\mathbb{Z}} (qM)_k$, where

$$(qM)_k = M_{k-1} \ (k\in\mathbb{Z}).$$

We call q the *grade-shift functor* on the category of graded $R(n)$-modules.

For $n \in \mathbb{Z}_{\ge 0}$ and $\beta \in Q_+$ such that $|\beta| = n$, we set

$$I^{\beta} = \{\nu = (\nu_1, \dots, \nu_n) \in I^n \,;\, \alpha_{\nu_1} + \dots + \alpha_{\nu_n} = \beta\},$$

and

$$R(\beta) = \bigoplus_{\nu\in I^{\beta}} R(n)e(\nu).$$

Note that for each $\beta \in \mathsf{Q}^+$, the element $e(\beta) := \sum_{\nu \in I^\beta} e(\nu)$ is a central idempotent of $R(n)$. The algebra $R(\beta)$ is called the *Khovanov-Lauda-Rouquier algebra at* β.

For a graded $R(m)$-module M and a graded $R(n)$-module N, we define the *convolution product* $M \circ N$ by

$$M \circ N = R(m+n) \underset{R(m)\otimes R(n)}{\otimes} (M \otimes N).$$

Let us denote by $R(\beta)$-proj (respectively, $R(\beta)$-gmod) the category of finitely generated graded projective (respectively, finite-dimensional over $\mathbf{k}$ graded) $R(\beta)$-modules. When $K(\mathrm{R}(\beta)\text{-proj})$ and $K(\mathrm{R}(\beta)\text{-gmod})$ denote the corresponding Grothendieck groups, the spaces

$$\bigoplus_{\beta \in \mathsf{Q}^+} K(R(\beta)\text{-proj}), \quad \bigoplus_{\beta \in \mathsf{Q}^+} K(R(\beta)\text{-gmod})$$

are $\mathbf{A}$-algebras with multiplications given by convolution products and $\mathbf{A}$-actions given by the grade-shift functor q.

A Khovanov-Lauda-Rouquier algebra *categorifies* the negative half of the corresponding quantum group. More precisely, we have the following theorem.

Theorem 1.4 ([16, 19])**.** *Let $U_q(\mathfrak{g})$ be the quantum group associated with a given Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ and $R = \bigoplus_{n \geq 0} R(n)$ be the Khovanov-Lauda-Rouquier algebra associated with the same Cartan datum and a matrix $(Q_{ij})_{i,j \in I}$ given in* (1.1)*. Then there exists an $\mathbf{A}$-algebra isomorphism*

$$U_{\mathbf{A}}^-(\mathfrak{g}) \xrightarrow{\sim} \bigoplus_{\beta \in \mathsf{Q}^+} K(R(\beta)\text{-proj}).$$

By duality, we have

$$U_{\mathbf{A}}^-(\mathfrak{g})^\vee \xrightarrow{\sim} \bigoplus_{\beta \in \mathsf{Q}^+} K(R(\beta)\text{-gmod}),$$

where $U_{\mathbf{A}}^-(\mathfrak{g})^\vee := \big\{ x \in U_q^-(\mathfrak{g})\,;\, (x, U_{\mathbf{A}}^-(\mathfrak{g})) \subset \mathbf{A} \big\}$.

The Khovanov-Lauda-Rouquier algebras also categorify the global bases in the following sense:

Theorem 1.5 ([21, 20])**.** *Assume that A is symmetric. Then under the isomorphism in Theorem 1.4, the lower global basis* (*respectively, upper global basis*) *corresponds to the set of isomorphism classes of indecomposable projective modules* (*respectively, the set of isomorphism classes of simple modules*)*.*

2. Quantum affine algebras and their representations

2.1. **Quantum affine algebras.**

In this section, we briefly review the representation theory of quantum affine algebras following [1, 14] and introduce a functor between the category of Khovanov-Lauda-Rouquier algebra modules and the category of quantum affine algebra modules. Hereafter, we take $\mathbb{C}(q)$ as the base field $\mathbf{k}$.

Let $I = \{0, 1, \dots, n\}$ be an index set and $A = (a_{ij})_{i,j\in I}$ be a generalized Cartan matrix of affine type; i.e., A is positive semidefinite of corank 1. Here 0 is chosen as the leftmost vertices in the tables in [11, pages 48, 49]. We take a Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ as follows.

The coweight lattice $P^\vee$ is given by

$$P^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d.$$

The element d is called the *scaling element*. We define the simple roots α_i's ($i \in I$) and the fundamental weights Λ_i's ($i \in I$) in the weight lattice $P := \mathrm{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z})$ as follows:

$$\alpha_i(h_j) = a_{ji},\ \alpha_i(d) = \delta_{0,i},\ \text{and}\ \Lambda_i(h_j) = \delta_{i,j},\ \Lambda_i(d) = 0.$$

We denote by $\Pi = \{\alpha_i \,;\, i \in I\}$ and $\Pi^\vee = \{h_i \,;\, i \in I\}$ the set of simple roots and the set of simple coroots, respectively.

Let us denote by $\mathfrak{g}$ the affine Kac-Moody algebra corresponding to the Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$. Consider the positive integers c_i's and d_i's determined by the conditions

$$\sum_{i=0}^{n} c_i a_{ij} = \sum_{i=0}^{n} a_{ji} d_i = 0 \quad \text{for all } j \in I,$$

and $\{c_0, c_1, \dots, c_n\}$, $\{d_0, d_1, \dots, d_n\}$ are relatively prime positive integers (see [11, Chapter 4]). Then the center of $\mathfrak{g}$ is 1-dimensional and is generated by the *canonical central element*

$$c = c_0h_0 + c_1h_1 + \cdots + c_nh_n$$

([11, Proposition 1.6]). Also it is known that the imaginary roots of $\mathfrak{g}$ are nonzero integral multiples of the *null root*

$$\delta = d_0\alpha_0 + d_1\alpha_1 + \cdots + d_n\alpha_n$$

([11, Theorem 5.6]). Note that $d_0 = 1$ if $\mathfrak{g} \neq A_{2n}^{(2)}$ and $d_0 = 2$ if $\mathfrak{g} = A_{2n}^{(2)}$. Note also that $c_0 = 1$ in all cases.

Now the weight lattice can be written as

$$P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n \oplus \mathbb{Z}d_0^{-1}\delta$$

(see [11, Chapter 4]). We have

$$\alpha_0 = \sum_{i=0}^{n} a_{ij}\Lambda_i + d_0^{-1}\delta, \quad \alpha_j = \sum_{i=0}^{n} a_{ij}\Lambda_i \text{ for } j = 1, \dots, n.$$

Let us denote by $U_q(\mathfrak{g})$ the quantum group associated with the affine Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$. We denote by $U_q'(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, f_i, K_i^{\pm 1} (i = 0, 1, \ldots, n)$ and it is called the *quantum affine algebra*. Hereafter we extend the base field $\mathbb{Q}(q)$ of $U_q'(\mathfrak{g})$ to $\mathbf{k} := \mathbb{C}(q)$ for convenience.

Set

$$P_{\mathrm{cl}}^\vee = \mathbb{Z}h_0 \oplus \cdots \oplus \mathbb{Z}h_n \subset P^\vee, \text{ and } \mathfrak{h}_{\mathrm{cl}} = \mathbb{Q} \otimes_{\mathbb{Z}} P_{\mathrm{cl}}^\vee \subset \mathfrak{h}.$$

Let $\mathrm{cl}\colon \mathfrak{h}^* \to (\mathfrak{h}_{\mathrm{cl}})^*$ be the projection thus obtained. Then $\mathrm{cl}^{-1}(0) = \mathbb{Q}\delta$. We denote the *classical weight lattice* $\mathrm{cl}(P)$ by P_{cl}. Set $\Pi_{\mathrm{cl}} = \mathrm{cl}(\Pi)$, and set $\Pi_{\mathrm{cl}}^\vee = \{h_0, \ldots, h_n\}$. Then $U_q'(\mathfrak{g})$ can be regarded as the quantum group associated with the quintuple $(A, P_{\mathrm{cl}}, \Pi_{\mathrm{cl}}, P_{\mathrm{cl}}^\vee, \Pi_{\mathrm{cl}}^\vee)$.

Set $\mathfrak{h}^{*0} = \{\lambda \in \mathfrak{h}^* \mid \lambda(c) = 0\}$, $\mathfrak{h}_{\mathrm{cl}}^{*0} = \mathrm{cl}(\mathfrak{h}^{*0})$ and $P_{\mathrm{cl}}^0 = \mathrm{cl}(P) \cap \mathrm{cl}(\mathfrak{h}^{*0})$. We call the elements of P_{cl}^0 by the *classical integral weight of level* 0. Let W be the *Weyl group* of $\mathfrak{g}$. It is the subgroup of $\mathrm{Aut}(\mathfrak{h}^*)$ generated by the simple reflections $\sigma_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$ for $i = 0, 1, \ldots, n$. Since $\delta(h_i) = \alpha_i(c) = 0$ for $i = 0, 1, \ldots, n$, there exists a group homomorphism $W \to \mathrm{Aut}(\mathfrak{h}_{\mathrm{cl}}^{*0})$. We denote the image by W_{cl}. Then W_{cl} is a finite group and it is isomorphic to the subgroup of W generated by $\sigma_1, \ldots, \sigma_n$.

A $U_q'(\mathfrak{g})$-module M is called an *integrable module* if M has a weight space decomposition

$$M = \bigoplus_{\lambda \in P_{\mathrm{cl}}} M_\lambda,$$

where $M_\lambda = \{u \in M \ ; \ q^h u = q^{\langle h, \lambda \rangle} u \text{ for all } h \in P_{\mathrm{cl}}^\vee\}$, and if the actions of e_i and f_i on M are locally nilpotent for any $i \in I$. In this paper, we mainly consider the category of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules. Let us denote this category by $\mathscr{C}$. The objects in this category are called of type 1 (for example, see [3]).

Definition 2.1. *Let u be a weight vector of weight $\lambda \in P_{\mathrm{cl}}$ of an integrable $U_q'(\mathfrak{g})$-module M. We call u* extremal, *if we can find vectors $\{u_w\}_{w \in W}$ satisfying the following properties:*

$$\begin{aligned}
&u_w = u \text{ for } w = e, \\
&\text{if } \langle h_i, w\lambda \rangle \geq 0, \text{ then } e_i u_w = 0 \text{ and } f_i^{(\langle h_i, w\lambda \rangle)} u_w = u_{s_i w}, \\
&\text{if } \langle h_i, w\lambda \rangle \leq 0, \text{ then } f_i u_w = 0 \text{ and } e_i^{(\langle h_i, w\lambda \rangle)} = u_{s_i w}.
\end{aligned}$$

Hence if such $\{u_w\}_{w \in W}$ exists, then it is unique and u_w has weight $w\lambda$. We denote u_w by $S_w u$.

For $\lambda \in P$, let us denote by $W(\lambda)$ the $U_q(\mathfrak{g})$-module generated by u_λ with the defining relation that u_λ is an extremal vector of weight λ (see [13]). This is in fact a set of infinitely many linear relations on u_λ.

Set $\varpi_i = \Lambda_i - c_i \Lambda_0 \in P^0$ for $i = 1, 2, \ldots, n$. Then $\{\mathrm{cl}(\varpi_i)\}_{i=1,2,\ldots,n}$ forms a basis of P_{cl}^0. We call ϖ_i a *level* 0 *fundamental weight*. As shown in [14], for each $i = 1, \ldots, n$ there exists a $U_q'(\mathfrak{g})$-module automorphism $z_i \colon W(\varpi_i) \to W(\varpi_i)$ which sends u_{ϖ_i} to $u_{\varpi_i + \mathsf{d}_i \delta}$,

where $\mathsf{d_i} \in \mathbb{Z}_{>0}$ denotes the generator of the free abelian group $\{m \in \mathbb{Z}\ ;\ \varpi_i + m\delta \in W\varpi_i\}$.

We define the $U_q'(\mathfrak{g})$-module $V(\varpi_i)$ by

$$V(\varpi_i) = W(\varpi_i)/(z_i - 1)W(\varpi_i).$$

It can be characterized as follows([1, Section 1.3]):

(1) The weights of $V(\varpi_i)$ are contained in the convex hull of $W_{\mathrm{cl}}\mathrm{cl}(\varpi_i)$.
(2) $\dim V(\varpi_i)_{\mathrm{cl}(\varpi_i)} = 1$.
(3) For any $\mu \in W_{\mathrm{cl}}\mathrm{cl}(\varpi_i) \subset P^0_{\mathrm{cl}}$, we can associate a nonzero vector u_μ of weight μ such that

$$u_{s_i\mu} = \begin{cases} f_i^{(\langle h_i,\mu\rangle)} u_\mu & \text{if } \langle h_i, \mu\rangle \geq 0, \\ e_i^{(-\langle h_i,\mu\rangle)} u_\mu & \text{if } \langle h_i, \mu\rangle \leq 0. \end{cases}$$

We call $V(\varpi_i)$ the *fundamental representation of* $U_q'(\mathfrak{g})$ *of weight* ϖ_i.

Let $-$ be an involution of a $U_q'(\mathfrak{g})$-module M satisfying $\overline{au} = \bar{a}\bar{u}$ for any $a \in U_q'(\mathfrak{g})$ and $u \in M$. We call such an involution a *bar involution*.

We say that a finite crystal B with weight in P^0_{cl} is a *simple crystal* if there exists $\lambda \in P^0_{\mathrm{cl}}$ such that $\#(B_\lambda) = 1$ and the weight of any extremal vector of B is contained $W_{\mathrm{cl}}\lambda$.

If a $U_q'(\mathfrak{g})$-module M has a bar involution, a crystal base with simple crystal graph, and a global base, then we say that M is a *good module* ([14, Section 8]). For example, the fundamental representation $V(\varpi_i)$ is a good $U_q'(\mathfrak{g})$-module. Any good module is an irreducible $U_q'(\mathfrak{g})$-module.

Let A be a commutative $\mathbf{k}$-algebra and let x be an invertible element of A. For an $A \otimes_{\mathbf{k}} U_q'(\mathfrak{g})$-module M, let us denote by $\Phi_x(M)$ the $A \otimes_{\mathbf{k}} U_q'(\mathfrak{g})$-module constructed in the following: there exists an A-linear bijection $\Phi_x\colon M \to \Phi_x(M)$ which satisfy

$$q^h\Phi_x(u) = \Phi_x(q^h u) \quad (h \in P^\vee_{\mathrm{cl}}), \quad e_i\Phi_x(u) = x^{\delta_{i,0}}\Phi_x(e_i u), \quad f_i\Phi_x(u) = x^{-\delta_{i,0}}\Phi_x(f_i u).$$

For invertible elements in x, y of A and $A \otimes_{\mathbf{k}} U_q'(\mathfrak{g})$-modules M, N, we have

$$\Phi_x\Phi_y(M) \simeq \Phi_{xy}(M)$$

by $\Phi_x(\Phi_y(u)) \leftrightarrow \Phi_{xy}(u)$, and

$$\Phi_x(M \otimes_{\mathbf{k}} N) \simeq \Phi_x(M) \otimes_{\mathbf{k}} \Phi_x(N)$$

by $\Phi_x(u \otimes v) \leftrightarrow \Phi_x(u) \otimes \Phi_x(v)$.

For an integrable $U_q'(\mathfrak{g})$-module M, the *affinization of* M is given by

$$M_{\mathrm{aff}} := \Phi_z(\mathbf{k}[z, z^{-1}] \otimes_{\mathbf{k}} M).$$

Note that by defining $q^d\Phi_z(z^n \otimes u) = q^{\langle d, n\delta\rangle}\Phi_z(z^n \otimes u)$ for $u \in M$, M_{aff} becomes a $U_q(\mathfrak{g})$-module. (We need a slight modification for $\mathfrak{g} = A^{(2)}_{2n}$.) For example, we have

$V(\varpi_i)_{\mathrm{aff}} \simeq \mathbf{k}[z_i^{1/\mathsf{d}_i}] \otimes_{\mathbf{k}[z_i]} W(\varpi_i)$, and hence if $\mathsf{d}_i = 1$, then $W(\varpi_i) \simeq V(\varpi_i)_{\mathrm{aff}}$ [14, Theorem 5.15].

For $a \in \mathbf{k}^\times$, we define $U_q'(\mathfrak{g})$-module M_a by

$$M_a := M_{\mathrm{aff}}/(z-a)M_{\mathrm{aff}} \simeq \Phi_a(M).$$

It is called the *evaluation module of M at a.*

2.2. R-matrices.

We recall the notion of the R-matrices of good modules following [14, Section 8].

Let M_1 and M_2 be good $U_q'(\mathfrak{g})$-modules. Set $(M_1)_{\mathrm{aff}} = \Phi_{z_1}(\mathbf{k}[z_1^{\pm 1}] \otimes M_1)$, $(M_2)_{\mathrm{aff}} = \Phi_{z_2}(\mathbf{k}[z_2^{\pm 1}] \otimes M_2)$, and let u_1 and u_2 be the dominant extremal weight vectors in M_1 and M_2, respectively.

Then there exists a $U_q'(\mathfrak{g})$-module homomorphism

$$R^{\mathrm{norm}}_{M_1,M_2} \colon (M_1)_{\mathrm{aff}} \otimes (M_2)_{\mathrm{aff}} \to \mathbf{k}(z_1,z_2) \otimes_{\mathbf{k}[z_1^{\pm1},z_2^{\pm1}]} \big((M_2)_{\mathrm{aff}} \otimes (M_1)_{\mathrm{aff}}\big),$$

satisfying

$$R^{\mathrm{norm}}_{M_1,M_2} \circ z_i = z_i \circ R^{\mathrm{norm}}_{M_1,M_2} \text{ for } i = 1,2 \tag{2.1}$$

and

$$R^{\mathrm{norm}}_{M_1,M_2}(u_1 \otimes u_2) = u_2 \otimes u_1$$

([14, Section 8]).

Let $d_{M_1,M_2}(u) \in \mathbf{k}[u]$ be a monic polynomial with the smallest degree such that the image of $d_{M_1,M_2}(z_1/z_2) R^{\mathrm{norm}}_{M_1,M_2}$ is contained in $(M_2)_{\mathrm{aff}} \otimes (M_1)_{\mathrm{aff}}$. We call $R^{\mathrm{norm}}_{M_1,M_2}$ the *normalized R-matrix* and d_{M_1,M_2} the *denominator of* $R^{\mathrm{norm}}_{M_1,M_2}$. Since $(M_1)_x \otimes (M_2)_y$ is irreducible for generic $x, y \in \mathbf{k}^\times$, we have

$$R^{\mathrm{norm}}_{M_2,M_1} \circ R^{\mathrm{norm}}_{M_1,M_2} = 1_{(M_1)_{\mathrm{aff}} \otimes (M_2)_{\mathrm{aff}}}. \tag{2.2}$$

It also satisfies the Yang-Baxter equation

$$(R^{\mathrm{norm}}_{M_1,M_2} \otimes 1) \circ (1 \otimes R^{\mathrm{norm}}_{M_1,M_3}) \circ (R^{\mathrm{norm}}_{M_2,M_3} \otimes 1) = (1 \otimes R^{\mathrm{norm}}_{M_2,M_3}) \circ (R^{\mathrm{norm}}_{M_1,M_3} \otimes 1) \circ (1 \otimes R^{\mathrm{norm}}_{M_1,M_2}). \tag{2.3}$$

The following fact is proved in [14, Proposition 9.3].

Lemma 2.2. *The zeroes of $d_{M_1,M_2}(z)$ belong to $\mathbb{C}[[q^{1/m}]]q^{1/m}$ for some $m \in \mathbb{Z}_{>0}$.*

Example 2.3. When $\mathfrak{g} = \widehat{\mathfrak{sl}}_N$, the normalized R-matrices for the fundamental representations are given as follows (see, for example, [6]):

$$R^{\mathrm{norm}}_{V(\varpi_k),V(\varpi_\ell)} = \sum_{0 \le i \le \min\{k,\ell\}} \prod_{s=1}^{i} \frac{1-(-q)^{|k-\ell|+2s} z}{z-(-q)^{|k-\ell|+2s}} \, P_{\varpi_{\max\{k,\ell\}+i} + \varpi_{\max\{k,\ell\}-i}},$$

where $z = z_1/z_2$ and P_λ denotes the projection from $V(\varpi_k) \otimes V(\varpi_\ell)$ to the direct summand $V(\lambda)$ as a $U_q(\mathfrak{sl}_N)$-module.

Note that $R^{\mathrm{norm}}_{V(\varpi_k),V(\varpi_\ell)}$ has simple poles at $z=(-q)^{|k-\ell|+2s}$ for $1 \le s \le \min\{k,\ell\}$.

2.3. **The action of $R^I(n)$ on $\widehat{V}^{\otimes n}$.** Let $\{V_s\}_{s\in\mathcal{S}}$ be a family of good $U_q'(\mathfrak{g})$-modules and let λ_s be a dominant extremal weight of V_s and v_s a dominant extremal weight vector in V_s of weight λ.

Let $\mathbb{T}=\mathbf{k}^\times$ and let I be a finite subset of $\mathcal{S}\times\mathbb{T}$. For each $i\in I$, let $X\colon I\to\mathbb{T}$ and $S\colon I\to\mathcal{S}$ be the maps defined by $i=(S(i),X(i))$.

For each $i,j\in I$, set

$$P_{ij}(u,v)=(v-u)^{d_{ij}},$$

where d_{ij} denotes the order of the zero of $d_{V_{S(i)},V_{S(j)}}(z_1/z_2)$ at $z_1/z_2=X(i)/X(j)$.

Let R^I be the Khovanov-Lauda-Rouquier algebra associated with

$$Q_{ij}(u,v)=P_{ij}(u,v)P_{ji}(v,u) \tag{2.4}$$

for $i,j\in I$.

Remark 2.4. Consider the quiver $\Gamma_I:=(I,\Omega)$ with the set of vertices I and the set of oriented edges Ω such that

$$\#\{h\in\Omega\,;\ s(h)=i,t(h)=j\}=d_{ij},$$

where $s(h)$ and $t(h)$ denote the source and the target of an oriented edge $h\in\Omega$.

Lemma 2.2 implies that $X(i)/X(j)\in\mathbb{C}[[q]]q$ if $d_{ij}>0$. Hence we obtain

$$\text{if } d_{ij}>0, \text{ then } d_{ji}=0.$$

Thus the quiver Γ_I has neither loops nor 2-cycles. The underlying unoriented graph of Γ_I gives a symmetric Cartan datum and the polynomials in (2.4) coincide with the ones used in [21] associated with the symmetric Cartan datum of Γ_I.

Set

$$\begin{aligned}\mathbb{P}_n &:= \bigoplus_{\nu\in I^n}\mathbf{k}[x_1,\dots,x_n]e(\nu),\\ \widehat{\mathbb{P}}_n &:= \bigoplus_{\nu\in I^n}\widehat{\mathcal{O}}_{\mathbb{T}^n,X(\nu)}e(\nu),\\ \widehat{\mathbb{K}}_n &:= \bigoplus_{\nu\in I^n}\widehat{\mathbb{K}}_\nu e(\nu),\end{aligned}$$

where

$$\widehat{\mathcal{O}}_{\mathbb{T}^n,X(\nu)}=\mathbf{k}[[X_1-X(\nu_1),\dots,X_n-X(\nu_n)]]$$

is the completion of the local ring of $\mathbb{T}^n$ at $X(\nu):=(X(\nu_1),\dots,X(\nu_n))$ and $\widehat{\mathbb{K}}_\nu$ is the field of quotients of $\widehat{\mathcal{O}}_{\mathbb{T}^n,X(\nu)}$.

Then we have

$$\mathbb{P}_n\hookrightarrow\widehat{\mathbb{P}}_n\hookrightarrow\widehat{\mathbb{K}}_n$$

as $\mathbf{k}$-algebras, where the first arrow is given by

$$x_k e(\nu) \mapsto (X(\nu_k)^{-1} X_k - 1) e(\nu).$$

Note that

$$\mathbf{k}[X_1^{\pm 1}, \dots, X_n^{\pm 1}] \subset \widehat{\mathcal{O}}_{\mathbb{T}^n, X(\nu)} \text{ for all } \nu \in I^n.$$

Let

$$\mathbf{k}[S_n] := \bigoplus_{w \in S_n} \mathbf{k} r_w$$

be the group algebra of S_n; i.e., the $\mathbf{k}$-algebra with the defining relations

$$\begin{aligned} & r_a^2 = 1 && a = 1, \dots, n-1 \\ & r_w r_{w'} = r_{ww'}, \\ & r_a r_{a+1} r_a = r_{a+1} r_a r_{a+1} && a = 1, \dots, n-2 \end{aligned} \tag{2.5}$$

where $r_a = r_{s_a}$ $(1 \le a < n)$.

The symmetric group S_n acts on $\mathbb{P}_n$, $\widehat{\mathbb{P}}_n$, $\widehat{\mathbb{K}}_n$ from the left and we have

$$\mathbb{P}_n \otimes \mathbf{k}[S_n] \hookrightarrow \widehat{\mathbb{P}}_n \otimes \mathbf{k}[S_n] \hookrightarrow \widehat{\mathbb{K}}_n \otimes \mathbf{k}[S_n]$$

as algebras. Here the algebra structure on $\widehat{\mathbb{K}}_n \otimes \mathbf{k}[S_n]$ is given by

$$r_w f = w(f) r_w \quad \text{for } f \in \widehat{\mathbb{K}}_n,\ w \in S_n. \tag{2.6}$$

Then $\widehat{\mathbb{K}}_n$ may be regarded as a right $\widehat{\mathbb{K}}_n \otimes \mathbf{k}[S_n]$-module by $a(f \otimes r_w) = w^{-1}(af)$ $(a, f \in \widehat{\mathbb{K}}_n$ and $w \in S_n)$.

Set

$$e(\nu)\tau_a = \begin{cases} e(\nu) r_a P_{\nu_a, \nu_{a+1}}(x_{a+1}, x_a) & \text{if } \nu_a \neq \nu_{a+1} \\ e(\nu)(r_a - 1)(x_a - x_{a+1})^{-1} & \text{if } \nu_a = \nu_{a+1}. \end{cases}$$

Then the subalgebra of $\widehat{\mathbb{K}}_n \otimes \mathbf{k}[S_n]$ generated by

$$e(\nu)\ (\nu \in J^n), \qquad x_a\ (1 \le a \le n), \qquad e(\nu)\tau_a\ (1 \le a \le n-1)$$

is isomorphic to the Khovanov-Lauda-Rouquier algebra $R^I(n)$ of degree n associated with $Q_{ij}(u,v) = P_{ij}(u,v) P_{ji}(v,u)$. [19, Proposition 3.12], [16, Theorem 2.5].

For each $\nu = (\nu_1, \dots, \nu_n) \in I^n$, we set

$$V_\nu = \Phi_{X_1}(\mathbf{k}[X_1^{\pm 1}] \otimes V_{S(\nu_1)}) \otimes \cdots \otimes \Phi_{X_n}(\mathbf{k}[X_n^{\pm 1}] \otimes V_{S(\nu_n)})$$

which is a $\mathbf{k}[X_1^{\pm 1}, \dots, X_n^{\pm 1}] \otimes U_q'(\mathfrak{g})$-module. Then we define

$$\begin{aligned} \widehat{V}^{\otimes n} &:= \bigoplus_{\nu \in I^n} \widehat{\mathcal{O}}_{\mathbb{T}^n, X(\nu)} \otimes_{\mathbf{k}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]} V_\nu e(\nu), \\ \widehat{V}_K^n &:= \widehat{\mathbb{K}}_n \otimes_{\mathbb{P}_n} \widehat{V}^{\otimes n}. \end{aligned} \tag{2.7}$$

For each $\nu \in I^n$ and $a = 1, \dots, n-1$, there exists a $U_q'(\mathfrak{g})$-module homomorphism

$$R^{\nu}_{a,a+1} \colon V_\nu \to \mathbf{k}(X_1, \dots, X_n) \otimes_{\mathbf{k}[X_1^{\pm 1}, \dots X_n^{\pm 1}]} V_{s_a(\nu)}$$

which is given by

$$v_1 \otimes \cdots \otimes v_a \otimes v_{a+1} \otimes \cdots \otimes v_n \mapsto v_1 \otimes \cdots \otimes R^{\mathrm{norm}}_{V_{S(\nu_a)}, V_{S(\nu_{a+1})}}(v_a \otimes v_{a+1}) \otimes \cdots \otimes v_n$$

for $v_k \in \Phi_{X_k}(V_{S(\nu_k)})$ $(1 \le k \le n)$.

It follows that

$$\begin{aligned}
&R^{\nu}_{a,a+1} \circ X_k = X_{s_a(k)} \circ R^{\nu}_{a,a+1} && \text{from (2.1)},\\
&R^{s_a(\nu)}_{a,a+1} \circ R^{\nu}_{a,a+1} = 1_{V_\nu} && \text{from (2.2)},\\
&R^{s_{a+1}s_a(\nu)}_{a,a+1} \circ R^{s_a(\nu)}_{a+1,a+2} \circ R^{\nu}_{a,a+1} = R^{s_a s_{a+1}(\nu)}_{a+1,a+2} \circ R^{s_{a+1}(\nu)}_{a,a+1} \circ R^{\nu}_{a+1,a+2} && \text{from (2.3)}.
\end{aligned}$$

Set $d_{\nu_a,\nu_{a+1}}(u) = d_{V_{S(\nu_a)}, V_{S(\nu_{a+1})}}(u)$. Then,

$$d_{\nu_a,\nu_{a+1}}(X_{a+1}/X_a) R^{\nu}_{a,a+1} \colon V_\nu \to V_{s_a(\nu)}.$$

The algebra $\widehat{\mathbb{K}}_n \otimes \mathbf{k}[S_n]$ acts on $\widehat{V}^n_K$ from the right, where

$$\begin{aligned}
e(\nu) r_a \colon & \widehat{\mathbb{K}}_\nu \otimes_{\mathbf{k}[X_1^{\pm 1}, \dots X_n^{\pm 1}]} V_\nu \\
& \to \widehat{\mathbb{K}}_{s_a(\nu)} \otimes_{\mathbf{k}(X_1, \dots, X_n)} \big(\mathbf{k}(X_1, \dots, X_n) \otimes_{\mathbf{k}[X_1^{\pm 1}, \dots X_n^{\pm 1}]} V_{s_a(\nu)}\big)
\end{aligned}$$

is given by

$$(f \otimes v) e(\nu) r_a = s_a(f) e(s_a(\nu)) \otimes R^{\nu}_{a,a+1}(v)$$

for $f \in \widehat{\mathbb{K}}_\nu$, $v \in \mathbf{k}(X_1, \dots, X_n) \otimes_{\mathbf{k}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]} V_\nu$. The subalgebra $\widehat{\mathbb{K}}_n$ acts by the multiplication. The relations (2.5) and (2.6) follow from the properties of normalized R-matrices and hence we have a well-defined action of the algebra $\widehat{\mathbb{K}}_n \otimes \mathbf{k}[S_n]$ on $\widehat{V}^n$. Since the normalized R-matrices are $U_q'(\mathfrak{g})$-module homomorphisms, the right action of $\widehat{\mathbb{K}}_n \otimes \mathbf{k}[S_n]$ commutes with the left action of $U_q'(\mathfrak{g})$ on $\widehat{V}^n$.

Theorem 2.5. *The subspace $\widehat{V}^n$ of $\widehat{V}^n_K$ is stable under the action of the subalgebra $R^I(n)$ of $\widehat{\mathbb{K}}_n \otimes \mathbf{k}[S_n]$. In particular, $\widehat{V}^n$ has a structure of $(U_q'(\mathfrak{g}), R^I(n))$-bimodule.*

Proof. It is obvious that $\widehat{V}^n$ is stable by the actions of $e(\nu)$ $(\nu \in I^n)$ and x_a $(1 \le a \le n)$. Thus it is enough to show that $\widehat{V}^n$ is stable under $e(\nu)\tau_a$ $(\nu \in I^n,\ 1 \le a < n)$.

Assume $\nu_a \neq \nu_{a+1}$. Then we have

$$\begin{aligned}
&e(\nu) r_a P_{\nu_a,\nu_{a+1}}(x_{a+1}, x_a) \\
=&e(\nu) r_a d_{\nu_a,\nu_{a+1}}(X_{a+1}/X_a)\Big(e(s_a(\nu)) \frac{P_{\nu_a,\nu_{a+1}}(x_{a+1}, x_a)}{d_{\nu_a,\nu_{a+1}}(X_{a+1}/X_a)}\Big) \\
=&e(\nu) r_a d_{\nu_a,\nu_{a+1}}(X_{a+1}/X_a)\Big(e(s_a(\nu)) \frac{(X(\nu_{a+1})^{-1}X_a - X(\nu_a)^{-1}X_{a+1})^{d_{\nu_a,\nu_{a+1}}}}{d_{\nu_a,\nu_{a+1}}(X_{a+1}/X_a)}\Big).
\end{aligned}$$

Since $d_{\nu_a,\nu_{a+1}}$ is the multiplicity of the zero of the polynomial $d_{\nu_a,\nu_{a+1}}(X_{a+1}/X_a)$ at $X_{a+1}/X_a = X(\nu_a)/X(\nu_{a+1})$, we have

$$\frac{(X(\nu_{a+1})^{-1}X_a - X(\nu_a)^{-1}X_{a+1})^{d_{\nu_a,\nu_{a+1}}}}{d_{\nu_a,\nu_{a+1}}(X_{a+1}/X_a)} \in \widehat{\mathcal{O}}_{\mathbb{T}^n, X(s_a(\nu))}.$$

It follows that

$$\begin{aligned}
&\Big(\widehat{\mathcal{O}}_{\mathbb{T}^n,X(\nu)} \otimes_{\mathbf{k}[X_1^{\pm 1},\dots X_n^{\pm 1}]} V_\nu\Big) e(\nu)\tau_a \\
=&\Big(\big(\widehat{\mathcal{O}}_{\mathbb{T}^n,X(\nu)} \otimes_{\mathbf{k}[X_1^{\pm 1},\dots X_n^{\pm 1}]} V_\nu\big) e(\nu) r_a\Big) P_{\nu_a,\nu_{a+1}}(x_{a+1}, x_a) \\
\subset&\Big(\widehat{\mathcal{O}}_{\mathbb{T}^n,X(s_a(\nu))} \otimes_{\mathbf{k}[X_1^{\pm 1},\dots X_n^{\pm 1}]} V_{s_a(\nu)}\Big) \frac{(X(\nu_{a+1})^{-1}X_a - X(\nu_a)^{-1}X_{a+1})^{d_{\nu_a,\nu_{a+1}}}}{d_{\nu_a,\nu_{a+1}}(X_{a+1}/X_a)} \\
\subset&\widehat{\mathcal{O}}_{\mathbb{T}^n,X(s_a(\nu))} \otimes_{\mathbf{k}[X_1^{\pm 1},\dots X_n^{\pm 1}]} V_{s_a(\nu)},
\end{aligned}$$

as desired.

Assume $\nu_a = \nu_{a+1}$. Then $R^{\mathrm{norm}}_{V_{S(\nu_a)},V_{S(\nu_a)}}$ does not have a pole at $X_a = X_{a+1}$ by Lemma 2.2. Since $\Phi_x(V_{S(\nu_a)}) \otimes \Phi_x(V_{S(\nu_a)})$ is irreducible for any $x \in \mathbf{k}^\times$, we obtain $R^{\mathrm{norm}}_{V_{S(\nu_a)},V_{S(\nu_a)}}|_{X_a = X_{a+1}} = \mathrm{id}$. Therefore, we have

$$\begin{aligned}
&\Big(\widehat{\mathcal{O}}_{\mathbb{T}^n,X(\nu)} \otimes_{\mathbf{k}[X_1^{\pm 1},\dots X_n^{\pm 1}]} V_\nu\Big) e(\nu)\tau_a \\
=&\Big(\widehat{\mathcal{O}}_{\mathbb{T}^n,X(\nu)} \otimes_{\mathbf{k}[X_1^{\pm 1},\dots X_n^{\pm 1}]} V_\nu\Big) e(\nu)(r_a - 1)(x_a - x_{a+1})^{-1} \\
=&\Big(\widehat{\mathcal{O}}_{\mathbb{T}^n,X(\nu)} \otimes_{\mathbf{k}[X_1^{\pm 1},\dots X_n^{\pm 1}]} V_\nu\Big) e(\nu) X(\nu_a)(r_a - 1)(X_a - X_{a+1})^{-1} \\
\subset&\widehat{\mathcal{O}}_{\mathbb{T}^n,X(\nu)} \otimes_{\mathbf{k}[X_1^{\pm 1},\dots X_n^{\pm 1}]} V_\nu,
\end{aligned}$$

as desired. □

Since $\widehat{V}^n$ is a $(U_q'(\mathfrak{g}), R^I(n))$-bimodule, we can construct the following functor:

$$\begin{aligned}
\mathcal{F}_n \colon R^I(n)\text{-gmod} &\to U_q'(\mathfrak{g})\text{-mod} \\
M &\mapsto \mathcal{F}_n(M) := \widehat{V}^n \otimes_{R^I(n)} M,
\end{aligned}$$

where $U_q'(\mathfrak{g})$-mod denotes the category of finite-dimensional $U_q'(\mathfrak{g})$-modules.

Theorem 2.6. *Let $M_1 \in R^I(n_1)$-gmod and $M_2 \in R^I(n_2)$-gmod. Then there exists a canonical isomorphism of $U_q'(\mathfrak{g})$-modules*

$$\mathcal{F}_n(M_1 \circ M_2) \simeq \mathcal{F}_{n_1}(M_1) \otimes \mathcal{F}_{n_2}(M_2),$$

where $n = n_1 + n_2$.

Proof. For each $\nu = (\nu_1, \dots, \nu_n) \in I^n$, set $\nu' = (\nu_1, \dots, \nu_{n_1})$ and $\nu'' = (\nu_{n_1+1}, \dots, \nu_n)$. Then we have an algebra homomorphism $\widehat{\mathcal{O}}_{\mathbb{T}^{n_1}, X(\nu')} \otimes \widehat{\mathcal{O}}_{\mathbb{T}^{n_2}, X(\nu'')} \to \widehat{\mathcal{O}}_{\mathbb{T}^n, X(\nu)}$. Moreover, for any finite-dimensional $\widehat{\mathcal{O}}_{\mathbb{T}^{n_1}, X(\nu')}$-module L_1 and any finite-dimensional $\widehat{\mathcal{O}}_{\mathbb{T}^{n_1}, X(\nu'')}$-module L_2, the induced morphism

$$L_1 \otimes L_2 \to \widehat{\mathcal{O}}_{\mathbb{T}^n, X(\nu)} \underset{\widehat{\mathcal{O}}_{\mathbb{T}^{n_1}, X(\nu')} \otimes \widehat{\mathcal{O}}_{\mathbb{T}^{n_2}, X(\nu'')}}{\otimes} (L_1 \otimes L_2)$$

is an isomorphism. Hence for any finite-dimensional $\mathbb{P}_{n_1}$-module L_1 and any finite-dimensional $\mathbb{P}_{n_2}$-module L_2, the induced morphism

$$(\widehat{V}^{n_1} \otimes \widehat{V}^{n_2}) \underset{\mathbb{P}_{n_1} \otimes \mathbb{P}_{n_2}}{\otimes} (L_1 \otimes L_2) \to \widehat{V}^n \underset{\mathbb{P}_{n_1} \otimes \mathbb{P}_{n_2}}{\otimes} (L_1 \otimes L_2)$$

is an isomorphism.

The module $\widehat{V}^n \otimes_{R^I(n)} (M_1 \circ M_2) \simeq \widehat{V}^n \otimes_{R^I(n_1) \otimes R^I(n_2)} (M_1 \otimes M_2)$ is the quotient of $\widehat{V}^n \otimes_{\mathbb{P}_{n_1} \otimes \mathbb{P}_{n_2}} (M_1 \otimes M_2)$ by the submodule generated by $va \otimes u - v \otimes au$ where $a \in R^I(n_1) \otimes R^I(n_2)$, $v \in \widehat{V}^n$, $u \in M_1 \otimes M_2$. A similar result holds also for $\big(\widehat{V}^{n_1} \otimes \widehat{V}^{n_2}\big) \otimes_{R^I(n_1) \otimes R^I(n_2)} (M_1 \otimes M_2)$. Thus we obtain the desired result

$$\big(\widehat{V}^{n_1} \otimes \widehat{V}^{n_2}\big) \otimes_{R^I(n_1) \otimes R^I(n_2)} (M_1 \otimes M_2) \simeq \widehat{V}^n \otimes_{R^I(n_1) \otimes R^I(n_2)} (M_1 \otimes M_2).$$

□

The following propositions are key ingredients for proving our main theorem.

Proposition 2.7 ([15, Corollary 2.9], [2, Theorem 4.6])**.** *If the quiver associated with $R^I(n)$ is of type A, D, E, then $R^I(n)$ has finite global dimension.*

Proposition 2.8. *Let $A \to B$ be a homomorphism of algebras. We assume the following conditions:*

(a) *B is a finitely generated projective A-module,*
(b) $\mathrm{Hom}_A(B, A)$ *is a projective B-module,*
(c) *the global dimension of B is finite.*

Then we have:

(i) *any B-module projective over A is projective over B,*
(ii) *any B-module flat over A is flat over B.*

Proof. Since the proof is similar, we give only the proof of (ii).

Let us denote by $\mathrm{flat.dim_A}\, M$ the flat dimension of an A-module M. By (a) we have

$$\mathrm{flat.dim_A}(M) \le \mathrm{flat.dim_B}(M)$$

for any B-module M.

By (b), $\mathrm{Hom}_A(B, A) \otimes_A L$ is a flat B-module if L is a flat A-module. Indeed, the functor $X \otimes_B \mathrm{Hom}_A(B, A)$ is exact in $X \in \mathrm{Mod}(A^{\mathrm{opp}})$ and hence $X \otimes_B \mathrm{Hom}_A(B, A) \otimes_A L$ is also exact in X.

On the other hand, for any A-module L, the canonical B-module homomorphism

$$\mathrm{Hom}_A(B, A) \underset{A}{\otimes} L \to \mathrm{Hom}_A(B, L), \qquad f \otimes s \longmapsto (B \ni b \mapsto f(b)s)$$

is an isomorphism by (a). Hence we conclude that $\mathrm{Hom}_A(B, L)$ is a flat B-module for any flat A-module L. It immediately implies that

$$\mathrm{flat.dim_B}\big(\mathrm{Hom}_A(B, L)\big) \le \mathrm{flat.dim_A}(L) \quad \text{for any } A\text{-module } L.$$

Now, let M be a B-module. Then there exists a canonical B-module homomorphism

$$\varphi_M \colon M \to \mathrm{Hom}_A(B, M)$$

given by $\varphi_M(x)(b) = bx$. It is evidently injective.

In order to prove the proposition, it is enough to show the following statement for any $d \ge 0$:

$$\text{for any } B\text{-module } M,\ \mathrm{flat.dim_A}(M) \le d \text{ implies } \mathrm{flat.dim_B}(M) \le d.$$

We shall show it by the descending induction on d. If $d \gg 0$, it is a consequence of (c). Let M be a B-module with $\mathrm{flat.dim_A}(M) \le d$. We have an exact sequence

$$0 \to M \xrightarrow{\varphi_M} \mathrm{Hom}_A(B, M) \to N \to 0.$$

Then $\mathrm{flat.dim_A}\big(\mathrm{Hom}_A(B, M)\big) \le \mathrm{flat.dim_B}\big(\mathrm{Hom}_A(B, M)\big) \le \mathrm{flat.dim_A}(M) \le d$. Hence we have $\mathrm{flat.dim_A}\, N \le d + 1$, which implies that $\mathrm{flat.dim_B}(N) \le d + 1$ by the induction hypothesis. Finally we conclude that $\mathrm{flat.dim_B}(M) \le d$. Thus the induction proceeds. □

Theorem 2.9. *If the quiver associated with $R^I(n)$ is of type A, D, E, then the functor $\mathcal{F}_n$ is exact.*

Proof. Let us apply Proposition 2.8 with $A = \mathbb{P}_n$ and $B = R^I(n)$. The conditions (a) and (b) are well-known, and (c) is nothing but Proposition 2.7. Therefore, since $\widehat{V}^n$ is a flat $\mathbb{P}_n$-module, it is a flat $R^I(n)$-module. □

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